

MUKAI FLOPS AND PLÜCKER TYPE FORMULAS FOR HYPER-KÄHLER MANIFOLDS

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ABSTRACT. We study the intersection theory of complex Lagrangian subvarieties inside holomorphic symplectic manifolds. In particular, we study their behaviour under Mukai flops and give a rigorous proof of the Plücker type formula for Legendre dual subvarieties written down by the second author before. Then we apply the formula to study projective dual varieties in projective spaces.

1. INTRODUCTION

From Weinstein's point of view [66, 67], everything in the symplectic world is a Lagrangian submanifold. The classical intersection theory of Lagrangian submanifolds is defined by perturbing Lagrangians to have transversal intersections. Fukaya, Oh, Ohta and Ono introduced quantum corrections in terms of counting holomorphic disks bounding Lagrangians and defined their Floer theory [28]. It was later generalized by Akaho and Joyce [1] to immersed Lagrangian submanifolds (see also Alston and Bao [2] for some exact immersed cases).

The marriage of (complex) algebraic geometry and symplectic geometry produces an interesting subject which concerns properties of hyper-Kähler manifolds (or general holomorphic symplectic manifolds). Hyper-Kähler manifolds, which have \mathbb{S}^2 -twistor family of complex structures, form a particular type of even dimensional Calabi-Yau manifolds [70] and are interesting in geometry [23, 35, 36, 48, 55, 58], string theory and gauge theory [71, 5, 13, 14, 46, 21, 40, 24, 16, 17, 18, 19, 20].

In hyper-Kähler geometry, complex Lagrangian submanifolds are natural objects and serve as an important source of special Lagrangian submanifolds defined by Harvey and Lawson [33]. For complex (algebraic) Lagrangians, their intersection theory can be defined by the normal cone construction [29] which does not perturb subvarieties to generic positions and has the advantage for actual computations. Meanwhile, as for quantum corrections (i.e. holomorphic disks), one can show that they do not exist for generic complex structures in the twistor family (at least when Lagrangians intersect cleanly, see e.g. [48]). Because of this, we will concentrate on the classical intersection numbers of complex Lagrangians in hyper-Kähler manifolds.

The main purpose of this article is to study the behaviour of intersection numbers of complex Lagrangian subvarieties (more generally, half-dimensional subvarieties) under a birational transformation called the Mukai flop [58] of projective hyper-Kähler manifolds (more generally, even dimensional projective manifolds).

Let M be a $2n$ -dimensional projective manifold containing a \mathbb{P}^n with $\mathcal{N}_{\mathbb{P}^n/M} \cong T^*\mathbb{P}^n$. We denote its *Mukai flop* along \mathbb{P}^n by

$$\phi : M \dashrightarrow M^+,$$

which is the composition of the blow up \widetilde{M} of M along \mathbb{P}^n and the blow down of \widetilde{M} along the exceptional locus in another ruling. Note that the exceptional locus blows down to a \mathbb{P}^n in M^+ which we denote by $(\mathbb{P}^n)^* \subseteq M^+$. If $C \subseteq M$ is a half-dimensional closed irreducible subvariety, we denote its *strict transformation*¹ by

$$C^\vee \triangleq \overline{\phi(C \setminus \mathbb{P}^n)} \subseteq M^+, \quad \text{if } C \neq \mathbb{P}^n, \\ (\mathbb{P}^n)^\vee \triangleq (-1)^n (\mathbb{P}^n)^* \subseteq M^+,$$

which satisfies the reflection property $(C^\vee)^\vee = C$. The behaviour of intersection numbers of half-dimensional subvarieties under Mukai flops is given by the following Plücker type formula.

Theorem 1.1. (*Theorem 3.4, see also Theorem 23 of [48]*)

Let $\phi : M \dashrightarrow M^+$ be a Mukai flop along \mathbb{P}^n ($n \geq 2$) between projective manifolds, then

$$C_1 \cdot C_2 + \frac{(C_1 \cdot \mathbb{P}^n)(C_2 \cdot \mathbb{P}^n)}{(-1)^{n+1}(n+1)} = C_1^\vee \cdot C_2^\vee + \frac{(C_1^\vee \cdot (\mathbb{P}^n)^*)(C_2^\vee \cdot (\mathbb{P}^n)^*)}{(-1)^{n+1}(n+1)}$$

¹The strict transformation is often referred as *Legendre transformation* if M is hyper-Kähler and C is its Lagrangian subvariety.

holds for n -dimensional closed irreducible subvarieties C_1, C_2 of M and their strict transformations C_1^\vee, C_2^\vee in M^+ .

This formula was first written down (without proof) by the second author in [48]. We will give it a rigorous proof in this paper and then apply it to the study of the geometry and topology of projective duality.

Now we explain why this formula is of Plücker type. We consider two irreducible subvarieties S_i ($i = 1, 2$) in \mathbb{P}^n and their projective dual $S_i^\vee \subseteq (\mathbb{P}^n)^*$ in the dual projective space [30]. The conormal varieties $C_{S_i} \subseteq T^*\mathbb{P}^n$ are in fact strict transformations of $C_{S_i^\vee}$ under Mukai flop $T^*\mathbb{P}^n \dashrightarrow T^*(\mathbb{P}^n)^*$, i.e. $C_{S_i^\vee} = (C_{S_i})^\vee$. When $S_1 \pitchfork S_2$, the intersection of C_{S_1} and C_{S_2} happens inside the zero section of $T^*\mathbb{P}^n$, we use Theorem 1.1 to deduce

Theorem 1.2. (Theorem 3.8, Proposition 3.9, 3.10)

Let S_1, S_2 be two closed irreducible subvarieties in \mathbb{P}^n ($n \geq 2$) which intersect transversally and the same holds true for their dual varieties S_1^\vee, S_2^\vee in $(\mathbb{P}^n)^*$. Then we have

$$(-1)^* \left(\chi(S_1 \cap S_2) - \frac{\chi(S_1, \text{Eu}([S_1])) \chi(S_2, \text{Eu}([S_2]))}{n+1} \right) = \chi(S_1^\vee \cap S_2^\vee) - \frac{\chi(S_1^\vee, \text{Eu}([S_1^\vee])) \chi(S_2^\vee, \text{Eu}([S_2^\vee]))}{n+1},$$

where $*$ = $\dim S_1 + \dim S_2 + \dim S_1^\vee + \dim S_2^\vee$, Eu is MacPherson's Euler obstruction and $\chi(-, \text{Eu}([-]))$ is the weighted Euler characteristic with respect to constructible function $\text{Eu}([-])$.

As corollaries, we could apply this formula to determine degrees of projective dual varieties (Corollary 3.17) which recovers Ernström's generalized Plücker formulas (see also [25, 26, 43, 56]). We also use it to determine degree zero Chern-Mather classes, dimensions of projective dual varieties (see Corollary 3.13 and 3.18). Because of these applications to the geometry of projective duality, we call the formula in Theorem 1.1 of Plücker type.

The proof of Theorem 1.1 is based on an equivalence $D^b(M) \cong D^b(M^+)$ of derived categories of coherent sheaves established by Kawamata [42] and Namikawa [59]. From this point of view, the above equivalence of categories could be regarded as a more general 'Plücker type formula'.

The content of this paper: In section 2, we recall the intersection theory of complex Lagrangian subvarieties. In section 3, using derived equivalences established by Kawamata and Namikawa, we deduce a Plücker type formula for intersection numbers of half-dimensional subvarieties in even dimensional projective manifolds under Mukai flops. We also apply it to deduce a Plücker type formula for projective dual varieties in projective spaces which has many applications to the geometry of projective duality. In the final section, we construct examples of complex Lagrangian subvarieties in hyper-Kähler manifolds, which are motivated by Donaldson-Thomas' higher dimensional gauge theories.

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2. INTERSECTION THEORY OF COMPLEX LAGRANGIANS IN HYPER-KÄHLER MANIFOLDS

In this section, we recall some standard facts about the intersection theory of complex Lagrangian subvarieties inside holomorphic symplectic manifolds.

2.1. Some basic notions from hyper-Kähler geometry. Let (M, g) be a hyper-Kähler manifold with complex structures I, J, K which satisfy the Hamilton relation

$$I^2 = J^2 = K^2 = IJK = -Id.$$

Then the twistor family of complex structures on M can be expressed as

$$\mathbb{S}^2 = \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1 \text{ and } a, b, c \in \mathbb{R}\}$$

Fixing a complex structure J , (M, g, J) is a Calabi-Yau manifold which corresponds to the embedding $Sp(n) \subseteq SU(2n)$. We denote the Kähler form by ω_J . The parallel form

$$\Omega_J = \omega_I - i\omega_K$$

defines a *holomorphic symplectic structure* on (M, J) . And the J -holomorphic volume form for the Calabi-Yau structure on (M, g, J) is just the top exterior power of Ω_J .

A J -complex Lagrangian submanifold $L \subseteq M$ is a half-dimensional submanifold with

$$\Omega_J|_L = 0^2.$$

Similar to the real case, if L is a complex Lagrangian submanifold, we have $\mathcal{N}_{L/M} \cong T^*L$. In particular, for any half-dimensional Fano ($c_1 > 0$) J -complex submanifold, one can identify its normal bundle and cotangent bundle (see e.g. [48]).

More generally, a *complex Lagrangian subvariety* $L \subseteq M$ is an irreducible locally closed³ subvariety whose smooth locus is a complex Lagrangian submanifold. The following result will be useful for the construction of compact complex Lagrangian subvarieties.

Proposition 2.1. *Let $L \subseteq M$ be a complex Lagrangian subvariety in an algebraic symplectic manifold. Then its (Zariski) closure $\overline{L} \subseteq M$ (with reduced structures) is also a complex Lagrangian subvariety.*

Proof. As L is locally closed, it is an open subvariety of \overline{L} . The irreducibility of L extends to \overline{L} . We are left to show its smooth locus is a complex Lagrangian submanifold. Since $T_x(\overline{L})_{sm}$ being a complex Lagrangian subspace (i.e. $\Omega_J|_{T_x(\overline{L})_{sm}} = 0$) is a closed condition among points $x \in (\overline{L})_{sm}$, thus $(\overline{L})_{sm}$ is also a complex Lagrangian submanifold (as $L_{sm} \subseteq (\overline{L})_{sm}$ is open and $(\overline{L})_{sm}$ is irreducible). \square

We refer to [34, 36, 38, 48, 50] for more details about hyper-Kähler geometry.

2.2. Intersection numbers of complex Lagrangians. Let L_1 and L_2 be two closed complex Lagrangian subvarieties in a projective hyper-Kähler manifold M . We have the following two types of intersection numbers (defined for any two half-dimensional subvarieties).

Topological intersection: $L_1 \cdot L_2 \triangleq \deg([L_1] \cup [L_2])$, where $[\cdot]$ denotes the Poincaré dual of the corresponding fundamental class [31].

Algebraic intersection: $L_1 \cdot L_2$ is the degree of their intersection product (defined by taking normal cone and applying the refined Gysin map [29]). These two approaches have their own advantages and coincide with each other (see for instance Corollary 19.2 [29]).

Next, we discuss the case when L_1 and L_2 are both smooth, following the work of Behrend and Fantechi [7], [8], [9]. The scheme theoretical intersection $X = L_1 \cap L_2$ then has a symmetric obstruction theory [7]

$$\varphi : E^\bullet \rightarrow \mathbb{L}_X,$$

which can be represented as $E^\bullet \simeq [T^*M|_X \rightarrow T^*L_1|_X \oplus T^*L_2|_X]$ [9]. Then Behrend's result gives

$$L_1 \cdot L_2 = \deg[X, E^\bullet]^{vir} = \chi(X, \nu_X),$$

where $[X, E^\bullet]^{vir} \in A_0(L_1 \cap L_2)$ is the virtual cycle [8], [51] of the perfect obstruction theory φ , ν_X is Behrend's constructible function of X defined as MacPherson's Euler obstruction of the intrinsic normal cone of X , and $\chi(X, \nu_X) = \sum_{n \in \mathbb{Z}} n \chi\{\nu_X = n\}$ is the weighted Euler characteristic with respect to ν_X . This then implies that for smooth complex Lagrangian submanifolds L_1, L_2 , their intersection number **depends only** on the scheme structure of the intersection $X = L_1 \cap L_2$. In particular, we could use weighted Euler characteristics as the definition of intersection numbers even when the intersection is non-compact.

2.3. Intersection cohomologies of complex Lagrangians. The symmetric obstruction theory on the scheme theoretical intersection $X = L_1 \cap L_2$ of two complex Lagrangian submanifolds in fact can be enhanced to a more refined structure called (-1) -shifted symplectic structure in the sense of Pantev, Töen, Vaquié and Vezzosi [62]. Joyce introduced d-critical loci [39] as classical truncations of derived schemes with (-1) -shifted symplectic structures and showed that for any oriented d-critical loci $(X, s, K_{X,s}^{1/2})$, there exists a perverse sheaf $\mathcal{P}_{X,s}^\bullet$ on X whose hypercohomology categorifies the weighted Euler characteristic of X , i.e.

$$\sum_i (-1)^i \mathbb{H}^i(X, \mathcal{P}_{X,s}^\bullet) = \chi(X, \nu_X).$$

The orientation $K_{X,s}^{1/2}$ in the Lagrangian intersection case corresponds to a choice of square roots of $K_{L_1}|_{X^{red}} \otimes K_{L_2}|_{X^{red}}$ over the reduced scheme X^{red} of X . In particular, if L_i ($i = 1, 2$) are *orientable* complex Lagrangian submanifolds (i.e. $\exists K_{L_i}^{1/2}$) [12], [15], $X = L_1 \cap L_2$ is orientable.

²Hitchin showed that this already implies L is a J -holomorphic submanifold of M [34].

³i.e. it is the open subset of a closed subset in the Zariski topology.

Other interesting works towards the categorification or quantization of complex Lagrangian intersections include works of Baranovsky and Ginzburg [4], Kapustin and Rozansky [40], Kashiwara and Schapira [41], etc.

Remark 2.2. Given two *orientable* complex Lagrangians L_1, L_2 in a hyper-Kähler manifold (with a fixed holomorphic symplectic form), they are real Lagrangians for a S^1 -family of real symplectic structures ω_θ . Now we have two Floer type cohomologies associated with them, i.e. Joyce's version: $\mathbb{H}^*(L_1 \cap L_2, \mathcal{P}_{L_1 \cap L_2}^\bullet)$;

Fukaya-Oh-Ohta-Ono's version: $HF_{\omega_\theta}^*(L_1, L_2) \triangleq HF_{\omega_\theta}^*(L_1, \phi(L_2))$ for a symplectic form ω_θ and a Hamiltonian diffeomorphism ϕ such that $L_1 \pitchfork \phi(L_2)$ ⁴.

When the intersection $L_1 \cap L_2$ is clean, $\mathcal{P}_{L_1 \cap L_2}^\bullet$ is a constant sheaf and Joyce's Floer cohomology is simply the singular cohomology $H^*(L_1 \cap L_2)$. Then there exists a spectral sequence

$$E_2 = H^*(L_1 \cap L_2) \Rightarrow HF_{\omega_\theta}^*(L_1, L_2)$$

converging to $HF_{\omega_\theta}^*(L_1, L_2)$ whose E_2 page is $H^*(L_1 \cap L_2)$ with certain coefficient in the Novikov field (see Theorem 6.1.4 of FOOO [28] for detail). In this set-up, for at most one exceptional $\theta \in [0, 2\pi)$, there is no J_θ -holomorphic disk bounding $L_1 \cup L_2$ ⁵, and the spectral sequence degenerates at E_2 page. It is an interesting question to extend this picture to singular intersection cases.

3. MUKAI FLOPS AND PLÜCKER TYPE FORMULAS

In this section, we study the behaviour of intersection numbers of complex Lagrangian subvarieties (more generally, half-dimensional subvarieties) inside projective hyper-Kähler manifolds (more generally, even dimensional projective manifolds) under the Mukai flop, which is summarized as a Plücker type formula. Then we apply it to projective dual varieties in projective spaces and give applications to their geometry and topology.

3.1. Mukai flops and derived equivalences. We recall the definition of Mukai flops and how derived categories of coherent sheaves behave under these birational transformations between projective manifolds of even dimensions (besides projective hyper-Kähler manifolds).

We start with a $2n$ -dimensional complex projective manifold M which contains an embedded \mathbb{P}^n such that $\mathcal{N}_{\mathbb{P}^n/M} \cong T^*\mathbb{P}^n$. We denote the blow up of M along \mathbb{P}^n by \widetilde{M} whose exceptional divisor $E = \mathbb{P}(T^*\mathbb{P}^n) \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^*$ is the incidence variety. By the adjunction formula, one has $\mathcal{N}_{E/\widetilde{M}} \cong \mathcal{O}(-1, -1)$. Thus \widetilde{M} admits a blow down $\widetilde{M} \rightarrow M^+$ to a projective manifold M^+ whose restriction to E is the projection $E \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^* \rightarrow (\mathbb{P}^n)^*$ to the second factor [37]. The birational transformation

$$\phi : M \dashrightarrow M^+$$

given by the composition of the above blow up and down is called *Mukai flop* along \mathbb{P}^n [57, 58, 37].

As normal bundles of \mathbb{P}^n and $(\mathbb{P}^n)^*$ are their cotangent bundles, M and M^+ admit blow down to a variety \overline{M} . We denote the fiber product $\widehat{M} = M \times_{\overline{M}} M^+$ which has canonical morphisms

$$(1) \quad \begin{array}{ccc} & \widehat{M} & \\ \pi \swarrow & & \searrow \pi^+ \\ M & & M^+ \end{array}$$

to M and M^+ . \widehat{M} is a normal crossing variety with two irreducible components \widetilde{M} and $\mathbb{P}^n \times (\mathbb{P}^n)^*$.

Theorem 3.1. (*Kawamata* [42], *Namikawa* [59])

If $n \geq 2$, the functor

$$\Psi \triangleq \mathbf{R}(\pi^+)_* \circ \mathbf{L}\pi^* : D^b(M) \rightarrow D^b(M^+)$$

is an equivalence of triangulated categories.

In particular, given two bounded complexes of sheaves \mathcal{E}_i^\bullet , $i = 1, 2$, we have isomorphisms

$$Ext_M^*(\mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet) \cong Ext_{M^+}^*(\Psi(\mathcal{E}_1^\bullet), \Psi(\mathcal{E}_2^\bullet)),$$

and an equality

$$(2) \quad \int_M \overline{ch}(\mathcal{E}_1^\bullet) \cdot ch(\mathcal{E}_2^\bullet) \cdot Td(M) = \int_{M^+} \overline{ch}(\Psi(\mathcal{E}_1^\bullet)) \cdot ch(\Psi(\mathcal{E}_2^\bullet)) \cdot Td(M^+),$$

by the Grothendieck-Hirzebruch-Riemann-Roch theorem (see [37]), where $\overline{ch} \triangleq \sum_k (-1)^k ch_k$.

⁴We use Novikov field as coefficient to kill torsion, so it coincides with Definition 6.5.39 of [28].

⁵See e.g. Lemma 13 of [48].

3.2. Plücker type formulas. Classical Plücker formulas relate degrees, numbers of double points and cusps of dual curves in \mathbb{P}^2 and its dual space $(\mathbb{P}^2)^*$ [31]. The purpose of this subsection is to generalize these formulas to higher dimensions based on equality (2) which is deduced from the derived equivalence in Theorem 3.1.

Definition 3.2. Let $\phi : M \dashrightarrow M^+$ be a Mukai flop along \mathbb{P}^n , and C be a half-dimensional closed irreducible subvariety in M . The *strict transformation* of C is

$$C^\vee \triangleq \overline{\pi^+(\pi^{-1}(C \setminus \mathbb{P}^n))} \subseteq M^+, \quad \text{if } C \neq \mathbb{P}^n,$$

$$(\mathbb{P}^n)^\vee \triangleq (-1)^n (\mathbb{P}^n)^* \subseteq M^+,$$

where $\pi : \widehat{M} \rightarrow M$, $\pi^+ : \widehat{M} \rightarrow M^+$ are the canonical morphisms in (1).

Remark 3.3. $(C^\vee)^\vee = C$ holds true for irreducible subvarieties.

A Plücker type formula for strict transformations is given as follows.

Theorem 3.4. (*Plücker type formula for Mukai flop*)

Let $\phi : M \dashrightarrow M^+$ be a Mukai flop along \mathbb{P}^n ($n \geq 2$) between projective manifolds, then

$$C_1 \cdot C_2 + \frac{(C_1 \cdot \mathbb{P}^n)(C_2 \cdot \mathbb{P}^n)}{(-1)^{n+1}(n+1)} = C_1^\vee \cdot C_2^\vee + \frac{(C_1^\vee \cdot (\mathbb{P}^n)^*)(C_2^\vee \cdot (\mathbb{P}^n)^*)}{(-1)^{n+1}(n+1)}$$

holds for n -dimensional closed irreducible subvarieties C_1, C_2 of M and their strict transformations C_1^\vee, C_2^\vee in M^+ .

Remark 3.5. For half-dimensional subvarieties in M not containing the \mathbb{P}^n , we can similarly define their strict transformations. Since $(C_1 \cup C_2)^\vee = C_1^\vee \cup C_2^\vee$ holds for such subvarieties, the above formula can be extended to (not necessarily irreducible) subvarieties in M which do not contain the \mathbb{P}^n .

Proof. We apply (2) to $\mathcal{O}_{C_1}, \mathcal{O}_{C_2}$, and then use the following Lemma 3.6 to determine Chern characters of $\Psi(\mathcal{O}_{C_i}), i = 1, 2$. \square

Lemma 3.6. Let $\Psi : D^b(M) \rightarrow D^b(M^+)$ be the equivalence in Theorem 3.1, then

$$ch(\Psi(\mathcal{O}_{\mathbb{P}^n})) = \pm[(\mathbb{P}^n)^*] + h.o.t.,$$

$$ch(\Psi(\mathcal{O}_C)) = [C^\vee] + \frac{\pm C \cdot \mathbb{P}^n - C^\vee \cdot (\mathbb{P}^n)^*}{(-1)^n(n+1)} [(\mathbb{P}^n)^*] + h.o.t.,$$

where C is a n -dimensional closed irreducible subvariety in M not containing the \mathbb{P}^n , $[C]$ denotes its Poincaré dual, and *h.o.t* stands for 'higher order terms'.

Proof. (i) As M and M^+ are isomorphic outside \mathbb{P}^n and $(\mathbb{P}^n)^*$, $supp(\Psi(\mathcal{O}_{\mathbb{P}^n})) \subseteq (\mathbb{P}^n)^*$. Then

$$ch(\Psi(\mathcal{O}_{\mathbb{P}^n})) = \alpha [(\mathbb{P}^n)^*] + h.o.t.$$

Applying (2) to $\mathcal{O}_{\mathbb{P}^n}$ and $\mathcal{O}_{\mathbb{P}^n}$, we obtain $\alpha^2 = 1$.

(ii) Away from $(\mathbb{P}^n)^* \subseteq M^+$, we have

$$supp(\Psi(\mathcal{O}_C)) \setminus (\mathbb{P}^n)^* = \pi^+(\pi^{-1}(C \setminus \mathbb{P}^n)) = C^\vee \setminus (\mathbb{P}^n)^*.$$

Since $supp(\Psi(\mathcal{O}_C))$ is closed, it then contains C^\vee (with multiplicity one) and

$$ch(\Psi(\mathcal{O}_C)) = [C^\vee] + \beta [(\mathbb{P}^n)^*] + h.o.t.$$

Applying (2) to \mathcal{O}_C and $\mathcal{O}_{\mathbb{P}^n}$, we obtain $\pm C \cdot \mathbb{P}^n = C^\vee \cdot (\mathbb{P}^n)^* + \beta \cdot (\mathbb{P}^n)^* \cdot (\mathbb{P}^n)^*$. \square

For hyper-Kähler manifold M and its Lagrangian subvariety C , the above strict transformation is often referred as Legendre transformation. The Plücker type formula then gives constraints to Legendre dual Lagrangians inside hyper-Kähler manifolds

Corollary 3.7. Let M be a projective hyper-Kähler manifold containing a \mathbb{P}^n ($n \geq 2$), then

$$(n+1)(\chi(C) - \chi(C^\vee)) = (C \cdot \mathbb{P}^n)^2 - (C^\vee \cdot (\mathbb{P}^n)^*)^2$$

holds if C is a compact complex Lagrangian submanifold such that C^\vee is also smooth.

Proof. Note that $\mathcal{N}_{C/M} \cong T^*C$ and $C \cdot C = (-1)^n \chi(C)$. Meanwhile, if C is a Lagrangian, its dual is also a Lagrangian (see e.g. [48]). \square

3.3. Applications to projective dual varieties. Given an irreducible subvariety S in \mathbb{P}^n , there is a notion of projective dual S^\vee of S in the dual projective space $(\mathbb{P}^n)^*$ [30]. The conormal variety $C_S = \overline{\mathcal{N}_{S_{sm}/\mathbb{P}^n}^*}$ of S is an irreducible complex Lagrangian subvariety in $T^*\mathbb{P}^n$ whose Legendre transformation is the conormal variety $C_{S^\vee} = \overline{\mathcal{N}_{S_{sm}^\vee/\mathbb{P}^n}^*}$ of the projective dual S^\vee [48]. As a consequence of Theorem 3.4, we obtain a Plücker type formula for dual varieties inside projective spaces.

Theorem 3.8. *Let S_1, S_2 be two closed irreducible subvarieties in \mathbb{P}^n ($n \geq 2$) which intersect transversally and the same holds true ⁶ for their dual varieties S_1^\vee, S_2^\vee in $(\mathbb{P}^n)^*$. Then we have*

$$C_{S_1} \cdot C_{S_2} + \frac{(C_{S_1} \cdot \mathbb{P}^n)(C_{S_2} \cdot \mathbb{P}^n)}{(-1)^{n+1}(n+1)} = C_{S_1^\vee} \cdot C_{S_2^\vee} + \frac{(C_{S_1^\vee} \cdot (\mathbb{P}^n)^*)(C_{S_2^\vee} \cdot (\mathbb{P}^n)^*)}{(-1)^{n+1}(n+1)},$$

where C_{S_i} 's are conormal varieties of S_i 's, $i = 1, 2$.

Proof. Let $M = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus T^*\mathbb{P}^n)$ be a compactification of $T^*\mathbb{P}^n$, and \overline{C}_{S_i} be the closure of C_{S_i} in M . As the intersection of S_1 and S_2 is transversal, $C_{S_1} \cap C_{S_2} = \overline{C}_{S_1} \cap \overline{C}_{S_2} \subseteq \mathbb{P}^n$ and $C_{S_i} \cap \mathbb{P}^n = \overline{C}_{S_i} \cap \mathbb{P}^n$. We interpret products in the above formula using algebraic intersections. By the intersection theory [29] (see also the following two propositions), we are then left to prove the formula for $\overline{C}_{S_i} \subseteq M$. As topological intersection numbers coincide with algebraic intersection numbers (see for instance, Corollary 19.2 of [29]), by Theorem 3.4, we are done. \square

As the intersection of S_1 and S_2 is transversal, the intersection number $C_{S_1} \cdot C_{S_2}$ depends only on the Euler characteristic of $S_1 \cap S_2$.

Proposition 3.9. *Let L_1, L_2 be two irreducible Lagrangian subvarieties inside a holomorphic symplectic manifold M with transversal and compact intersection, then*

$$L_1 \cdot L_2 = (-1)^{\dim(L_1 \cap L_2)} \chi(L_1 \cap L_2).$$

When applied to Theorem 3.8, we have

$$C_{S_1} \cdot C_{S_2} = (-1)^{\dim(S_1 \cap S_2)} \chi(S_1 \cap S_2).$$

Proof. As $L_1 \pitchfork L_2$, there exists smooth neighbourhoods of $L_1 \cap L_2$ inside both L_i , $i = 1, 2$ and we can assume L_1, L_2 are smooth without loss of generality. By the standard excess intersection theory (see [9], [29]),

$$L_1 \cdot L_2 = c_{top}(E) \cap [L_1 \cap L_2],$$

where E is the excess bundle which fits into the exact sequence

$$0 \rightarrow T(L_1 \cap L_2) \rightarrow TL_1|_{L_1 \cap L_2} \oplus TL_2|_{L_1 \cap L_2} \rightarrow TM|_{L_1 \cap L_2} \rightarrow E \rightarrow 0.$$

The holomorphic symplectic form on M induces an isomorphism of this sequence to its dual. Thus $E \cong T^*(L_1 \cap L_2)$. \square

The intersection number $(C_{S_i} \cdot \mathbb{P}^n)$ in Theorem 3.8 is also an intrinsic invariant of S_i , $i = 1, 2$.

Proposition 3.10. (MacPherson [54], Behrend [7])

Let S be a closed irreducible subvariety of \mathbb{P}^n . We denote $c_0^M(S) \triangleq (-1)^{\dim S} (C_S \cdot \mathbb{P}^n)$.

Then it is the degree zero Chern-Mather class of S . Furthermore,

$$c_0^M(S) = \chi(S, Eu([S])),$$

where Eu is the Euler obstruction of S and $\chi(S, Eu([S])) \triangleq \sum_{n \in \mathbb{Z}} n \chi\{Eu([S]) = n\}$ is the weighted Euler characteristic with respect to the integer-valued constructible function $Eu([S])$.

Remark 3.11. If S is smooth, $Eu(S) = 1$ and $c_0^M(S) = \chi(S)$.

The following example shows the Plücker type formula for dual varieties is nontrivial.

Example 3.12. Let $S_1 \cong S_2 \cong \mathbb{P}^1$ be two transversal intersecting lines in \mathbb{P}^2 , and S_1^\vee, S_2^\vee are two different points in $(\mathbb{P}^2)^*$. The Plücker type formula in Theorem 3.8 gives

$$1 - \frac{4}{3} = S_1 \cap S_2 + \frac{\chi(\mathbb{P}^1) \cdot \chi(\mathbb{P}^1)}{-3} = 0 + \frac{\chi(pt) \cdot \chi(pt)}{-3} = -\frac{1}{3},$$

which shows $C_{S_1} \cdot C_{S_2} \neq C_{S_1^\vee} \cdot C_{S_2^\vee}$ in general.

By applying Theorem 3.8 to the case when $S_1 = S$ and $S_2 = \mathbb{P}^{n-k-1}$ with $0 \leq k \leq \text{codim}(S^\vee) - 1$, we can determine Chern-Mather classes of singular varieties.

⁶By Proposition 1.3 of [30], S_1^\vee, S_2^\vee are automatically irreducible.

Corollary 3.13. *Let $S \subseteq \mathbb{P}^n$ be a closed irreducible subvariety such that there exists a linear subspace $\mathbb{P}^{n-k-1} \subseteq \mathbb{P}^n$ with $\mathbb{P}^{n-k-1} \pitchfork S$ and $0 \leq k \leq \text{codim}(S^\vee) - 1$, then*

$$c_0^M(S^\vee) = (-1)^{\dim S + \dim S^\vee + n + 1} \left(\frac{n-k}{k+1} c_0^M(S) - \frac{n+1}{k+1} \chi(S \cap \mathbb{P}^{n-k-1}) \right)$$

Proof. Note that for generic $\mathbb{P}^{n-k-1} \subseteq \mathbb{P}^n$, $(\mathbb{P}^{n-k-1})^* \pitchfork S^\vee$ and $(\mathbb{P}^{n-k-1})^* \cap S^\vee = \emptyset$. \square

In particular, this recovers classical Plücker formulas for dual curves in \mathbb{P}^2 .

Corollary 3.14. *(Classical Plücker formula)*

Let $S_1 \subseteq \mathbb{P}^2$ be an irreducible curve with at most node and cusp singularities and the same holds true for the dual curve S_1^\vee .⁷ Suppose $S_2 \cong \mathbb{P}^1$ be a line in \mathbb{P}^2 intersecting S_1 transversally⁸, then

$$c_0^M(S_1^\vee) = 3\deg(S_1) - 2c_0^M(S_1),$$

where $c_0^M(S_1) \triangleq (-1)^{\dim S_1} C_{S_1} \cdot \mathbb{P}^n = -d^2 + 3d + 2\delta + 3\kappa$, and δ (resp. κ) denotes the number of nodes (resp. cusps) in S_1 .

Moreover, the above formula is equivalent to the classical Plücker formula⁹

$$d^\vee = d^2 - d - 2\delta - 3\kappa,$$

where d^\vee denotes the degree of the dual curve S_1^\vee .

Proof. From Corollary 3.13, $c_0^M(S_1^\vee) = 3\deg(S_1) - 2c_0^M(S_1)$, then we have

$$(3) \quad \begin{aligned} 2d^2 - 3d - 4\delta - 6\kappa &= -(d^\vee)^2 + 3d^\vee + 2\delta^\vee + 3\kappa^\vee, \\ -d^2 + 3d + 2\delta + 3\kappa &= 2(d^\vee)^2 - 3d^\vee - 4\delta^\vee - 6\kappa^\vee. \end{aligned}$$

Eliminating $(d^\vee)^2$ terms, we obtain $d^\vee = d^2 - d - 2\delta - 3\kappa$. The converse part is similar. \square

Remark 3.15. From the invariance of geometric genus for dual curves, we obtain

$$\frac{1}{2}(d-1)(d-2) - \delta - \kappa = \frac{1}{2}(d^\vee-1)(d^\vee-2) - \delta^\vee - \kappa^\vee.$$

Combined with (3), we get the second Plücker formula

$$\kappa^\vee = 3d(d-2) - 6\delta - 8\kappa$$

for plane curves (see for instance, Proposition 2.5 [30]).

When $S^\vee \cap (\mathbb{P}^{n-k-1})^* \neq \emptyset$ in Corollary 3.13, we have the following example.

Example 3.16. (Chern-Mather class of Beauville-Donagi's Pfaffian hypersurface)

Let $X^{13} \subseteq \mathbb{P}^{14}$ be the Pfaffian hypersurface of degree 3 in Beauville-Donagi [6], $\mathbb{P}^5 \subseteq \mathbb{P}^{14}$ be a generic linear subspace which intersects X^{13} transversally¹⁰ along a smooth cubic 4-fold. Inside the dual space $(\mathbb{P}^{14})^*$, we have $(X^{13})^\vee \cong Gr(2, 6)$, $(\mathbb{P}^5)^\vee \cong \mathbb{P}^8$, which intersect transversally along a K3 surface $S = (X^{13})^\vee \cap (\mathbb{P}^5)^\vee$. The Plücker type formula in Theorem 3.8 gives

$$\chi(X^{13} \cap \mathbb{P}^5) + \frac{c_0^M(X^{13}) \cdot \chi(\mathbb{P}^5)}{-15} = \chi(S) + \frac{\chi(Gr(2, 6)) \cdot \chi(\mathbb{P}^8)}{-15},$$

i.e. $c_0^M(X^{13}) = 30$.

By applying Theorem 3.8 to the case when $S_1 = S$ and $S_2 = \mathbb{P}^{n-\text{codim}(S^\vee)-1}$, one can determine the degree of dual varieties, which recovers Ernström's generalized Plücker formulas (see also [25, 26, 43, 56]).

Corollary 3.17. *Let $S \subseteq \mathbb{P}^n$ be a closed irreducible subvariety such that there exists a linear subspace $\mathbb{P}^{n-k-1} \subseteq \mathbb{P}^n$ with $\mathbb{P}^{n-k-1} \pitchfork S$ and $0 \leq k \leq \text{codim}(S^\vee) - 1$, then*

$$\deg(S^\vee) = (-1)^{\dim S + l + 1} \left(\frac{l-k}{k+1} c_0^M(S) + \chi(S \cap \mathbb{P}^{n-l-1}) - \frac{l+1}{k+1} \chi(S \cap \mathbb{P}^{n-k-1}) \right),$$

where $l = \text{codim} S^\vee$.

Proof. For generic $\mathbb{P}^{n-k-1} \subseteq \mathbb{P}^n$ with $0 \leq k \leq \text{codim}(S^\vee)$, $(\mathbb{P}^{n-k-1})^* \pitchfork S^\vee$, and the assumption ensures that there exists $\mathbb{P}^{n-\text{codim}(S^\vee)-1} \subseteq \mathbb{P}^n$ with $\mathbb{P}^{n-\text{codim}(S^\vee)-1} \pitchfork S$, then we apply Theorem 3.8 to $(S_1 = S, S_2 = \mathbb{P}^{n-k-1})$ and $(S_1 = S, S_2 = \mathbb{P}^{n-\text{codim}(S^\vee)-1})$ individually. Combining them and eliminating terms with $C_{S^\vee} \cdot (\mathbb{P}^n)^*$, we obtain the formula. \square

⁷This should be true generically, see [30].

⁸Such S_2 always exists as $\dim(S_1^{\text{sing}}) < \text{codim} S_2$.

⁹See Chapter 2.4 of [31] or Proposition 2.5 of [30].

¹⁰The intersection happens inside the smooth loci of X^{13} .

Combining Corollary 3.13, 3.17, we have the following criterion to detect dimensions of dual varieties.

Corollary 3.18. *Let $S \subseteq \mathbb{P}^n$ be a closed irreducible subvariety such that there exists a linear subspace $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ with $\mathbb{P}^{n-1} \pitchfork S$, then for any $0 \leq k \leq \text{codim}(S^\vee) - 1$,*

$$kc_0^M(S) = (k+1)\chi(S \cap \mathbb{P}^{n-1}) - \chi(S \cap \mathbb{P}^{n-k-1}),$$

which becomes a strict inequality exactly when $k = \text{codim}(S^\vee)$.

Proof. When $0 \leq k \leq \text{codim}(S^\vee) - 1$, we apply Theorem 3.8 to $(S_1 = S, S_2 = \mathbb{P}^{n-1})$ and $(S_1 = S, S_2 = \mathbb{P}^{n-k-1})$ and eliminate terms with $c_0^M(S^\vee)$. When $k = \text{codim}(S^\vee)$, we get a strict inequality as $\deg(S^\vee) > 0$. \square

We apply Theorem 3.8 to the situation when S_2 is a smooth quadric hypersurface, in which case its dual variety is again a smooth quadric hypersurface.

Corollary 3.19. *Let $S \subseteq \mathbb{P}^n$ be a closed irreducible subvariety such that there exists a quadric hypersurface Q intersecting S transversally and the same holds true for Q^\vee, S^\vee , then*

$$\chi(S \cap Q) - \left(1 - \frac{1 + (-1)^n}{2(n+1)}\right)c_0^M(S) = (-1)^{\dim S + \dim S^\vee} \left(\chi(S^\vee \cap Q^\vee) - \left(1 - \frac{1 + (-1)^n}{2(n+1)}\right)c_0^M(S^\vee) \right)$$

Proof. We apply Theorem 3.8 to $(S_1 = \mathbb{P}^{n-1}, S_2 = Q)$ and obtain $\chi(Q_n) = n + 1 + \frac{1 + (-1)^n}{2}$ for quadric hypersurface $Q_n \subseteq \mathbb{P}^{n+1}$, then apply it to $(S_1 = S, S_2 = Q)$. \square

4. APPENDIX ON EXAMPLES OF COMPLEX LAGRANGIANS FROM HIGHER DIMENSIONAL GAUGE THEORIES

The purpose of this section is to construct examples of complex Lagrangian subvarieties inside holomorphic symplectic manifolds following Donaldson-Thomas' work on higher dimensional gauge theories and their TQFT structures (see for instance [24], [64]).

We take a smooth anti-canonical divisor S of a complex projective 3-fold Y , and consider a moduli space \mathfrak{M}_Y of stable holomorphic bundles on Y with fixed Chern classes. To make sense of the restriction morphism,

$$r : \mathfrak{M}_Y \rightarrow \mathfrak{M}_S$$

to a moduli space \mathfrak{M}_S of stable sheaves on S , we recall the following criterion.

Theorem 4.1. (Flenner [27])

Let $(X, \mathcal{O}_X(1))$ be a complex n -dimensional normal projective variety with $\mathcal{O}_X(1)$ very ample. We take \mathcal{F} to be a $\mathcal{O}_X(1)$ -slope semi-stable torsion-free sheaf of rank r . d and $1 \leq c \leq n-1$ are integers such that

$$[\binom{n+d}{d} - cd - 1]/d > \deg(\mathcal{O}_X(1)) \cdot \max\left(\frac{r^2 - 1}{4}, 1\right).$$

Then for a generic complete intersection $Y = H_1 \cap \dots \cap H_c$ with $H_i \in |\mathcal{O}_X(d)|$, $\mathcal{F}|_Y \triangleq \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is $\mathcal{O}_X(1)|_Y$ -slope semi-stable on Y .

Remark 4.2. For $X = \mathbb{P}^3$, $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^3}(1)$ is very ample. We take $c = 1$, $d = 5$, then any semi-stable sheaf on X with $\text{rank} \leq 5$ remains semi-stable when restricted to a generic quartic $K3$ surface inside X .

Assuming conditions in Theorem 4.1 are satisfied, we obtain a morphism

$$r : \mathfrak{M}_Y \rightarrow \mathfrak{M}_S,$$

whose deformation obstruction theory is described by the following exact sequence.

Lemma 4.3. *Let $E \in \mathfrak{M}_Y$ be a stable bundle, and S be connected. Then there is a long exact sequence,*

$$\begin{aligned} 0 \rightarrow H^{0,1}(Y, \text{End}E \otimes K_Y) \rightarrow H^{0,1}(Y, \text{End}E) \rightarrow H^{0,1}(S, \text{End}E|_S) \rightarrow \\ \rightarrow H^{0,2}(Y, \text{End}E \otimes K_Y) \rightarrow H^{0,2}(Y, \text{End}E) \rightarrow 0. \end{aligned}$$

Proof. We tensor $0 \rightarrow \mathcal{O}_Y(-S) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_S \rightarrow 0$ with $\text{End}E$ and take its cohomology. \square

It was observed by Donaldson-Thomas [24] (in fact first by Tyurin [64]) that the transpose of the above sequence with respect to Serre duality pairings on Y and S remains the same, and $r(\mathfrak{M}_Y)$ will be a complex Lagrangian submanifold of \mathfrak{M}_S provided that $H^{0,2}(Y, \text{End}E) = 0$.

Example 4.4. (Li-Qin's example, Proposition 5.4 of [53])

Let S be a ($K3$) generic hyperplane section of type $(2, 3)$ in $Y = \mathbb{P}^1 \times \mathbb{P}^2$, and $L_m = \mathcal{O}_Y(1, m)$ be a polarization. Then the moduli space \mathfrak{M}_Y of rank 2, L_m -stable bundles on Y with Chern classes $c_1 = (\epsilon_1, \epsilon_2)$, $c_2 = (-1, 1) \cdot (\epsilon_1 + 1, \epsilon_2 - 1)$ is isomorphic to \mathbb{P}^k , where $(\epsilon_1, \epsilon_2) = (0, 1)$ or $(1, 0)$ or $(1, 1)$, and $k = (5 + 6\epsilon_1 - 3\epsilon_2 - 3\epsilon_1\epsilon_2)$ respectively.

Furthermore, if $\frac{2(2-\epsilon_2)}{2+\epsilon_1} < m < \frac{2(2-\epsilon_2)}{\epsilon_1}$, the restriction map

$$\mathfrak{M}_Y \hookrightarrow \mathfrak{M}_S$$

to a moduli space \mathfrak{M}_S of $L_m|_S$ -(semi)stable rank 2 torsion-free sheaves on S with Chern classes $c_1|_S, c_2|_S$ is an imbedding of a complex Lagrangian into a compact hyper-Kähler manifold.

In particular, if $(\epsilon_1, \epsilon_2) = (0, 1)$, for $m \geq 2$, we have $\mathfrak{M}_Y \cong \mathbb{P}^2$ which is **not spin** and is not an *orientable complex Lagrangian* based on Joyce's definition (see e.g. [12], [39] or Definition 1.16 of [15]). However, one could still categorify complex Lagrangian intersection $\mathfrak{M}_Y \cap \mathfrak{M}_Y \subseteq \mathfrak{M}_S$, where the cohomology theory is $H^*(\mathfrak{M}_Y, \mathbb{C})$ (see e.g. [9]).

The main difficulty of extending the above construction to general cases is that the restriction morphism is not well-defined for general stable sheaves. However, for ideal sheaves of subschemes, say I_Z 's, the restriction map is well-defined if Z 's are normal to the divisor $S \subseteq Y$, i.e. $\text{Tor}^1(\mathcal{O}_Z, \mathcal{O}_S) = 0$ ¹¹. Then such ideal sheaves form an open subspace U of the moduli scheme of ideal sheaves on Y which has a restriction morphism $r : U \rightarrow \text{Hilb}^*(S)$ to a Hilbert scheme of points on S . Then it is interesting to know when $\overline{\text{Im}(r)} \subseteq \text{Hilb}^*(S)$ is a complex Lagrangian subvariety.

Proposition 4.5. *Let L be a connected smooth complex quasi-projective variety, M be an algebraic symplectic manifold, and $r : L \rightarrow M$ be an algebraic morphism whose underlying complex analytic map is an imbedding of a complex Lagrangian submanifold. Then the (Zariski) closure $\overline{\text{Im}(r)}$ in M is a complex Lagrangian subvariety.*

Proof. The morphism r factors through a morphism $r : L \rightarrow \text{Im}(r)_{\text{sch}}$ to the scheme theoretic image of r , which is the 'smallest' closed subset of M containing the set $\text{Im}(r)$. As L is reduced, the scheme theoretic image of r coincides with the closure $\overline{\text{Im}(r)}$ (with reduced structures) of the image of r (see e.g. 8.3.A of [65]). As L and $\overline{\text{Im}(r)}$ are both algebraic (they are closed subvarieties of algebraic varieties), we can find a subset $U \subseteq \text{Im}(r)$ which is dense and open in $\overline{\text{Im}(r)}$ by Chevalley's lemma (see e.g. Lemma 2 of [63]). Then we have a morphism $r : r^{-1}(U) \rightarrow U$ whose underlying complex analytic map is an isomorphism. By Serre's GAGA principle (Proposition 9 of [63]), it is also an isomorphism between algebraic schemes. The connectedness of L implies that $r^{-1}(U)$, U are both irreducible, so is $\overline{\text{Im}(r)}$. Since U is an irreducible locally closed complex Lagrangian submanifold of M , by Proposition 2.1, $\overline{U} = \overline{\text{Im}(r)} \subseteq M$ is a complex Lagrangian subvariety. \square

We apply the above proposition to construct compact complex Lagrangian subvarieties.

Example 4.6. (Generic quartics in \mathbb{P}^3)

Let S be a generic quartic surface in $Y = \mathbb{P}^3$ as its anti-canonical divisor.

(i) We take the primitive curve class $[H] \in H_2(Y, \mathbb{Z})$. Ideal sheaves of curves representing this class have Chern character $c = (1, 0, -PD([H]), 1)$. We consider the moduli space $I_1(Y, [H])$ of such ideal sheaves on Y and $I_1(Y, [H]) \cong \text{Gr}(2, 4)$. As S is a projective $K3$ surface and contains only one primitive rational curve, denoted by C_0 [71, 5, 14, 21], we obtain a well-defined restriction morphism

$$\begin{aligned} r : I_1(Y, [H]) \setminus \{I_{C_0}\} &\rightarrow \text{Hilb}^4(S), \\ r : I_C &\mapsto I_C|_S, \end{aligned}$$

which is injective with smooth image. By direct calculations, for any $I_C \in I_1(Y, [H])$, we have

$$\text{Ext}_Y^1(I_C, I_C) \cong \mathbb{C}^4, \quad \text{Ext}_Y^{i \geq 2}(I_C, I_C) = 0.$$

Then the injective restriction map determines a complex Lagrangian submanifold

$$\text{Gr}(2, 4) \setminus \{pt\} \subseteq \text{Hilb}^4(S).$$

Its closure $\overline{\text{Im}(r)} \subseteq \text{Hilb}^4(S)$ is a complex Lagrangian subvariety by Proposition 4.5.

(ii) We take the degree 2 curve class $[2H] \in H_2(Y, \mathbb{Z})$, and ideal sheaves of curves representing this class have Chern character $c = (1, 0, -PD([2H]), 3)$. We consider the moduli space $I_3(Y, [2H])$

¹¹See the work of Li and Wu [52], [69].

of such ideal sheaves, and $I_3(Y, [2H])$ is a \mathbb{P}^5 -bundle over \mathbb{P}^3 ¹². As S contains only one rational curve of degree 2 [71, 46, 68, 44, 61] similarly as before, we have a well-defined restriction morphism

$$\begin{aligned} r : I_3(Y, [2H]) \setminus \{pt\} &\rightarrow \text{Hilb}^8(S), \\ r : I_C &\mapsto I_C|_S, \end{aligned}$$

which is injective with smooth image. By direct calculations, for any $I_C \in I_3(Y, [2H])$, we have

$$\text{Ext}_Y^1(I_C, I_C) \cong \mathbb{C}^8, \quad \text{Ext}_Y^{i \geq 2}(I_C, I_C) = 0.$$

Then the injective restriction map determines a complex Lagrangian submanifold

$$I_3(Y, [2H]) \setminus \{pt\} \subseteq \text{Hilb}^8(S)$$

and a compact complex Lagrangian subvariety $\overline{\text{Im}(r)} \subseteq \text{Hilb}^8(S)$.

Remark 4.7. One could consider $K3$ surfaces as anti-canonical divisors of other projective 3-folds and give more examples of complex Lagrangians inside hyper-Kähler manifolds.

For instance, we take a generic degree $1 \leq d \leq 4$ hypersurface $Y \subseteq \mathbb{P}^4$ and consider the primitive class $[H] \in H_2(Y, \mathbb{Z})$. Ideal sheaves of curves representing this class have Chern character $c = (1, 0, -PD([H]), \frac{3-d}{2})$, and we denote their moduli space by $I_{\frac{3-d}{2}}(Y, [H])$. By Theorem 4.3 in Chapter V of [45], $I_{\frac{3-d}{2}}(Y, [H])$ is a smooth connected projective variety of dimension $(5-d)$. By Chapter V, 4.4 of [45] and direct calculations, for any $I_C \in I_{\frac{3-d}{2}}(Y, [H])$,

$$\text{Ext}_Y^1(I_C, I_C) \cong \mathbb{C}^{5-d}, \quad \text{Ext}_Y^{i \geq 2}(I_C, I_C) = 0.$$

Then the restriction morphism to the Hilbert scheme of a generic anti-canonical divisor $S \subseteq Y$,

$$\begin{aligned} r : I_{\frac{3-d}{2}}(Y, [H]) \setminus \{pt\} &\rightarrow \text{Hilb}^{5-d}(S), \\ r : I_C &\mapsto I_C|_S, \end{aligned}$$

determines a complex Lagrangian submanifold

$$I_{\frac{3-d}{2}}(Y, [H]) \setminus \{pt\} \subseteq \text{Hilb}^{5-d}(S)$$

and a compact complex Lagrangian subvariety $\overline{\text{Im}(r)} \subseteq \text{Hilb}^{5-d}(S)$.

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¹²See Remark 2.2.7 of [22] for reference.

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